Dynamically Reconstructed Proof Visualizations in Real Analysis

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Abstract

This paper discusses one approach for teaching proof in a real analysis course that makes use of a contemporary dynamic technology. The dynamically reconstructed proof visualizations discussed in this paper are intended both to make abstract ideas in real analysis more visually dynamic and also to provide an environment for students to interact with the particular givens and arguments in a claim in order to foster insight and comprehension about its proof.

Proof is really important. But teaching proof-based courses is hard (e.g., [2], [5]). Consider the intermediate value theorem (IVT) in a real analysis course. Notably, the value we are looking to prove exists is a "static" real number, $c \in [a, b]$. A typical proof approach utilizes a nested intervals argument, which begins with a rule for constructing a desirable infinite sequence of nested intervals, which imposes a "dynamic" perspective on this "static" value. Then, limited by ink and paper (or chalk and board), this "dynamic" construction process is captured by a "static" image - i.e, Figure 1, from Abbott's (2015) textbook ([1]), is a typical visual that accompanies the exposition. The proof proceeds from there. It is no wonder students find proof-based courses difficult: a singular example is used as the visualization to prove a general claim, based on a imposing a dynamic argument depicted by a static image.

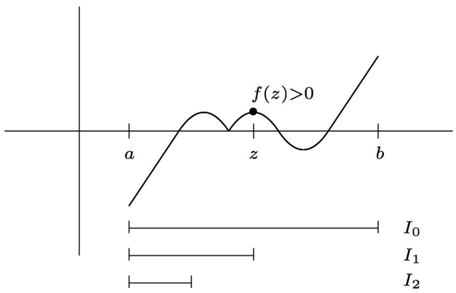

Figure 1: Visual of the nested interval proof approach for the IVT [1]

The purpose in this paper is to share one approach for teaching proof in a real analysis course that incorporates contemporary technology, and to exemplify it via two examples. Especially since proof methods in analysis often utilize infinite, dynamic processes that are encapsulated by static notation and symbols, I have used a dynamic geometry technology (GeoGebra) to create what I will refer to as *dynamically reconstructed proof visualizations*. Much use of dynamic technology (e.g., [4]) is aimed at helping students observe phenomenon, make conjectures, and understand a mathematical *claim*; in this paper, however, the use of dynamic technology is not necessarily for understanding a *claim*, but rather for understanding the *proof* of a claim. We adapt Leung's ([4]) epistemic model for task design with this goal of *proof comprehension* in mind. Now, visual representations, themselves, can be controversial with respect to proof. One the one hand, reliance on visuals can often be a barrier to rigorous proof; on the other hand, visuals often foster insight about proof, as well as about the meaning of mathematical objects in general. In this context, my use of dynamically reconstructed proof visualizations are not intended as a substitute for the proofs themselves; rather, they are a pedagogical aid and accompaniment to help students comprehend (and interact with) the proofs - an additional tool to help unpack their underlying nuances, processes, and arguments. Indeed, their goal is more about communicating and clarifying meaning than about formal verification. Note that, although I have used this approach in my own teaching and anecdotal evidence suggests that these approaches help students' learning of proofs, the purpose of this paper is to share and exemplify the technological approach, not to elaborate on supporting evidence from students. The paper is organized in a way — especially the different "variants" described - that attempts to clarify how the dynamic approach appeared to influenced students' understanding about proofs; but future research studies are needed to substantiate more definitive claims.

1 Dynamic Technology and Proofs

In general, the term "dynamic technology" is used to convey that users are able to dynamically manipulate and create objects. Dynamic manipulation means that the interactions are direct (e.g., a user points at an object and drags it, rather than writes lines of code or syntax) and change happens continuously in real time; the ability to create means that users create objects beginning from a blank slate and with a given set of tools. GeoGebra is a contemporary dynamic geometry technology - open source and available for free (http://www.geogebra.org). In GeoGebra, users have access, for example, to (Euclidean) geometric tools, such as the ability to create points, parallel/perpendicular lines, and circles (even planes). These familiar tools allow users to construct triangles, rectangles, bisectors, etc. However, once constructed, these objects are also dynamic, where manipulating any of the independent points changes and updates the object in real time, allowing, for example, one to construct the three perpendicular bisectors of a triangle and, by moving the vertices of the original triangle, see that the three lines will always be concurrent no matter the triangle. The available tools in GeoGebra, however, can easily branch into other areas of mathematics, including algebra, probability, statistics, etc. Dynamic technologies, including GeoGebra, have become a relatively mainstream technology in secondary mathematics education; and, for the most part, research indicates that these technologies can be used in ways that enhance learning about mathematics (e.g., [3]). One reason for this is that dynamic technologies provide a link between the *general* and the *specific*.

Linking the general claim being made with the specific examples that both instantiate and substantiate (or refute) the claim is a *particular* challenge with proofs and proof comprehension. Dynamic software, given its affordances, might be helpful. For proving, claims to be proved consist of both (i) given condition(s) and (ii) implication(s); proofs consist of particular generalizable processes and/or arguments (often consisting of various steps) that warrant linking the condition(s) and the implication(s). The claims, though, are about *general*, or *abstract*, objects - where these general objects can be instantiated by *specific* examples. For example, consider the general claim of the Intermediate Value Theorem (IVT): its conditions are (i) $f : [a, b] \to \mathbb{R}$, (ii) f is continuous, and (iii) $L \in \mathbb{R}$ such that $f(a) < L < f(b)$ (or vice versa); its implication is $\exists c \in [a, b]$ such that $f(c) = L$. But $f, [a, b], L$, etc., are abstract objects; specific examples of each of these are needed to instantiate, to exemplify, to *visualize*, etc., the proof of the claim. Notably, specific examples for $f, [a, b], L$ could either fulfill or violate the conditions; both are important for proof comprehension. Having visualizations be dynamic also affords some pedagogical advantages and opportunities that would not be possible otherwise. One advantage of a dynamic visualization is the ability to interact with some of the various abstract parts in conjunction with a concrete example. Rather than relying on a single example or illustration, students can engage with the visuals in a meaningful manner by altering and considering other kinds of examples as well.

In describing task design with dynamic software, especially for learning about mathematics, Leung ([4]) describes an epistemic model that consists of three modes: 1) Establishing Practices; 2) Critical Discernment; and 3) Establishing Situated Discourses. Establishing practices is about getting a feel for how to interact with the objects within the dynamic environment (e.g., "dragging"); critical discernment is about making important mathematical observations about one's interaction with the objects (e.g., perpendicular bisectors are always concurrent); and establishing situated discourse is about providing justification that explains one's observations. Because learning about and understanding specific proofs is not quite the same as learning about and understanding claims, I adapted these three modes in order to be more specific to proof. Before I describe some of these specifications (in the next section), I elaborate on two additional comments with respect to thinking about these modes within the context of real analysis. First, the notion of "dragging" (establishing practices) feels slightly different in this context. Although it is relatively easy to "drag" through a triangle's vertex the space of possibilities (i.e., $\mathbb{R}^2$) to consider the thousands of specific possible triangles, in real analysis the objects themselves are more abstract — which makes it difficult, for example, to meaningfully "drag" through the space of all possible real-valued functions. Second, technology itself has some challenges with respect to visualization. For example, not all functions can be accurately portrayed (e.g., there is no way to plot Dirilecht's function), so no technology can possibly include all the cases one might want to consider. Similarly, any computer technology is necessarily discrete. That is, in a course about real numbers and infinite processes, there will be moments at which things might break down. Nonetheless, the potential advantages with the possibility of dynamic manipulations and visual interactions in the area of proof comprehension are still compelling, despite these limitations.

- Establishing Practices: 1) Change the specific givens of the proof visualization (e.g., a statement about $[a, b]$ might be about $[2, 5]$ or $[1, 10]$); 2) Reconstruct the proof with the specific givens (e.g., reconstruct each step of the proof)
- Critical Discernment: Observe each step in the proof as related to the proof visualization
- Establishing Situated Discourses: 1) Generalize the proof process across all examples; 2) Recognize the role of each of the givens

2 Dynamically Reconstructed Proof Visualizations

So what do I mean by *dynamically reconstructed proof visualizations* as a means to aid proof comprehension in a real analysis course? Using the tools of GeoGebra, I created a pre-constructed environment to serve as the dynamic visualization of a proof. In other words, although GeoGebra as a dynamic technology includes beginning from a blank slate, I used the tools of GeoGebra to design an environment for students to interact with the proofs, and students then used these pre-constructed files. Two assumptions about these pre-constructed environments were operating in the design process. The first is that the aim of a dynamically reconstructed proof visualization is not just convincing a student about the truth or falsehood of a *claim*, but rather intentionally mimicking some of a proof's processes, arguments, ideas, etc., in a manner intended to foster insight and comprehension about the *proof of a claim*. The second is that the pre-constructed environments were intentionally created to allow students to interact with them. That is, although they could be used simply as a visual accompaniment to an instructor's lecture, they have been created with another purpose: letting students manipulate them on their own. (Of course, proper guidance and scaffolding by an instructor is always an important component of helping students be mathematically productive in their explorations.) The second is in accord with Leung's ([4]) model for task design; the first meant that some specifications about how this model worked for proofs and proof comprehension were needed.

In this context around proofs and proof comprehension, my adaptation of Leung's ([4]) model took on the following characteristics. *Establishing practices* meant that interactions would allow students to (i) change the specific examples used to instantiate the general objects that are part of the given conditions of the claim, and (ii) reconstruct the argument and steps of the proof on the specific examples of the given conditions. *Critical discernment* meant that the pre-constructed environment (i) aided students' abilities to observe and interact with each step of the proof's processes and/or arguments. *Establishing situated discourses* meant fostering opportunities for students to (i) generalize the proof's argument and its validity given the conditions, and (ii) recognize the necessity of each given condition and where in the proof's processes and/or arguments each given condition is utilized. This model for task design meant that: the given conditions of a claim are both explicit within the pre-constructed environment and the specific examples can be altered without disrupting the argument; and the particular processes and/or arguments in a proof can be repeated and rerun on the specified example, providing opportunities to consider why and in what ways the argument may fail by violating the given conditions.

In what follows, I provide examples of two different dynamically reconstructed proof visualizations for two fundamental ideas in a real analysis course: i) the Intermediate Value Theorem; and ii) the Algebraic Limit Theorem for the sum of two sequences based on the $\epsilon - N$ definition.

2.1 Intermediate Value Theorem

The Intermediate Value Theorem (IVT) is a crucial — and not obvious [6] — theorem in a real analysis course. It is stated explicitly to familiarize readers with the particular notation used: *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. If* L *is a real number satisfying* $f(a) < L < f(b)$ *[or* $f(a) > L > f(b)$*], then there exists a point* $c \in [a, b]$ *where* $f(c) = L$.

The dynamically reconstructed proof visualization in this paper uses the relatively common approach of a nested intervals argument, and presumes a corresponding definition of a real number.

(Throughout this discussion, without loss of generality, presume $f(a) < L < f(b)$.) For the sake of brevity, I do not provide the complete proof in this text, but summarize a few overall aspects of the proof, which takes on two distinct stages. The first stage is a *construction* process: that is, the proof describes a general process for constructing a sequence of nested intervals of the form $[a_n, b_n]$. The second is a *proof* about the construction: namely, that the constructed sequence of nested intervals contains the desired element - a point c such that $f(c) = L$. In my experience, students have difficulty following the infinite construction process and both differentiating it from and coordinating it with the proof. Notably, because of the abstract nature of the nested intervals, students have difficulty conceptualizing the sequence in relation to a specific example as well as some of the nuances that exist. I briefly mention a few critical parts to the proof. The first is the use of the midpoint, z_k, of an interval, I_k, in the construction of the next (nested) interval, I_{k+1}: if $f(z_k) < L$, then $a_{k+1} = z_k$ (and $b_{k+1} = b_k$); if $f(z_k) \geq L$, then $b_{k+1} = z_k$ (and $a_{k+1} = a_k$). I note that this construction process is dependent on being able to evaluate the function at an arbitrary midpoint - which, ultimately, requires the given function to be defined on all of $[a, b]$. The second is one line in the proof: "Since $a_n \to c$ and $b_n \to c$, then $\lim f(a_n) = f(c)$ and $\lim f(b_n) = f(c)$." This is the line in the proof that relies on the *continuity* of the function f — one whose domain is also an interval of real numbers.

2.2 An initial look at the dynamic visualization.

The pre-constructed GeoGebra file (https://www.geogebra.org/m/NNn3ed48) was created so as to mimic the nested intervals construction process, as well as to provide visuals and specific indicators that complement the proof. First, the specific given conditions of the claim have been included — the function f (initially, a hand-drawn arbitrary function), the interval $[a, b]$ (initially, $[1, 10]$), and the value of L (initially, the special case of $L = 0$). Each of these specific values/examples can be modified. One might also note that the statement of the IVT has been specified to this particular example, and that a button showing the possible range of L values, between $f(a)$ and $f(b)$, has been included. These have been done to help students unpack the meaning of the claim in the IVT, as well as to connect the general statement to the specific example. Second, a brief outline of the construction process and the claim are there, along with a slider that allows students to dynamically "play through" the overarching nested intervals argument with the specific givens by dragging the slider, n. As students walk through the argument (Figure 2), the midpoint of each interval, z, is visualized, as is the creation of the exact sequence of nested intervals described in the general proof. In addition, two buttons are included that are intended to help portray significant aspects of the proof: that the sequences (a_k) and (b_k) both approach c, and that the sequences $(f(a_k))$ and $(f(b_k))$ both approach L. Indeed, the zooming features in GeoGebra (which are, to my knowledge, not accessible in the online environment but are accessible if one downloads the file) are handy - they provide an opportunity to sense what is happening as the length of the nested intervals gets increasingly small. (I do note that no computer graphic can give a completely precise visualization of an interval of real numbers, or a function on such an interval, because computers, however precise they may be, operate using discrete principles — thus, zooming will, at some point, become a limitation and a liability. However, the visuals, including the ability to dynamically zoom in and out on the function, can foster productive insight about the generality of the construction process.)

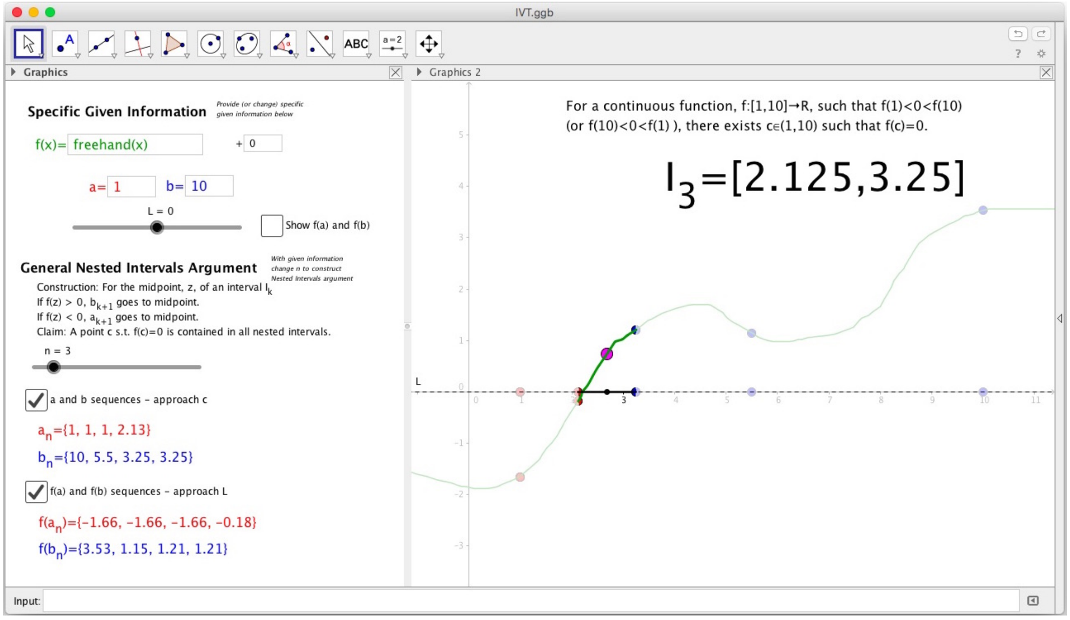

Figure 2: "Playing through" the Nested Intervals proof of the IVT

2.3 Possible explorations to develop proof comprehension through dynamically reconstructed proof visualizations.

As mentioned previously, one of the productive things about a dynamic visualization of a proof is that it affords exploring some of the various nuances and implications of the claim and the argument. A relatively trivial example includes changing the values of the initial interval $[a, b]$. Indeed, altering f to be another continuous function is also, arguably, somewhat trivial. A little less trivial, perhaps, is changing the value of L, which shows a new sequence of nested intervals based on whether $f(z)$ is above or below the line, $y = L$. Rather than focus on these alterations, all of which could (and perhaps should) be explored, I focus on three variants that help emphasize the utility of a dynamically reconstructed proof visualization as it relates to aiding proof comprehension.

Variant 1. What if there are multiple points c where $f(c) = L$? To accomplish this, one can simply change the value of L with the original arbitrary function f to be at a location where there are multiple such solutions to $f(c) = L$ in the interval $[a, b]$. In particular, why I find this to be a valuable variant to explore has to do with: i) making a clarification about the claim; and ii) making a clear implication about the nested interval construction process. In particular, the claim of the IVT is that *there exists* such a point c - not that such a point c is unique. Providing an example that demonstrates this nuance helps clarify a central idea of the IVT's claim. Second, by providing such an example in a dynamic environment, students also get a sense that the nested interval construction process, in fact, can only hone in to exactly *one* of these multiple possible points, c, depending on the initial interval. From the given information, the specific construction for the sequence of nested intervals cannot identify *any* of the other points c where $f(c) = L$ — only, and exactly, *one* of them.

Variant 2. What if f is discontinuous? Considering such a case is particularly valuable for proof comprehension, as it makes more explicit in the proof process where continuity becomes central to the claim and in the proof. By changing the function f to a step function (i.e., I typed

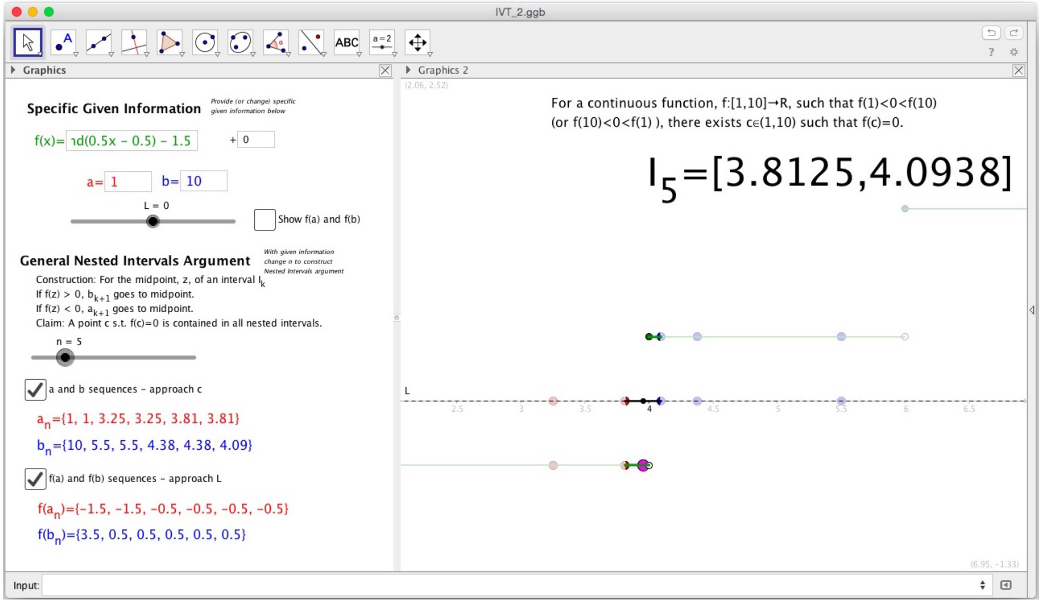

Figure 3: Breaking the continuity assumption in the IVT

"`round(0.5x-0.5)-1.5`" into the input bar for the function), and walking through the proof, at least three things become clear: i) the nested interval construction process works perfectly well for a discontinuous function defined on $[a, b]$; ii) both sequences (a_n) and (b_n) still approach the same value c; and iii) the problem is that the sequences $(f(a_n))$ and $(f(b_n))$ do not necessarily approach the same value, $f(c) = L$, since the function is discontinuous. In Figure 3, this becomes evident as the sequence of $(f(a_n))$ values tend toward -0.5 whereas the sequence of $(f(b_n))$ values tends toward 0.5, despite the fact that both (a_n) and (b_n) approach the same value, 4. That is, the line in the proof claiming that $\lim f(a_n) = f(c)$ and $\lim f(b_n) = f(c)$ is no longer valid. Indeed, it is precisely this line in the proof — which comes at a relatively late stage in the overall scheme of things — that necessitates the continuity of f in the claim. (The issue of continuity in the IVT is also of broader interest: exploring the more subtle idea about whether the intermediate value property implies continuity (which it does not) would also be interesting, although beyond the scope of the current manuscript.)

Variant 3. What if f is not defined on all of $[a, b]$? Another facet of an analysis course is clarifying that continuity, defined on a point-by-point basis, is intricately linked to a function's domain. Many students are surprised to find out that the definition of continuity in an analysis class would claim that the function $f(x) = 1/x$ is a continuous function on its presumed domain, $x \neq 0$, despite the vertical asymptote at $x = 0$. Now, if this function is modified to include a particular value for $f(0)$, then f would be considered discontinuous because of the fundamental discontinuity at $x = 0$. However, this brings to light another notable presumption about the IVT: that the entire interval $[a, b]$ is part of the function's domain. By exploring a translation of $1/x$ (i.e., one could type "`1/(x-3.25)`" into the input bar for the function), the necessity of the function being defined on the entire interval becomes apparent: namely, the construction process for the nested intervals stalls because there is no way to evaluate a particular value $f(z)$. When a function is undefined at the midpoint - in this case at $x = 3.25$ — there is no way to determine the next interval in the sequence of nested intervals.

Each of these three variants has led to an important insight about the IVT and its proof. It has clarified some of the major assumptions necessary for the theorem to hold — and thus the proof — as well as some of the implications of the nested intervals construction process and argument. Dynamically reconstructed proof visualizations become compelling for precisely this reason — they allow students to explore some of the ramifications of the particular proof method on specific cases to understood more clearly the necessity of the particular given conditions of the IVT, as well as the specific aspects of the proof that depend upon these assumptions.

2.4 Algebraic Limit Theorem for Sequences

The algebraic properties of limits, which frequently begin with sequences in a real analysis course, are essential for further developments of concepts such as functions, derivatives, and integrals. I address one — the sum property for limits: *If* $(a_n) \to a$ *and* $(b_n) \to b$*, then* $(a_n + b_n) \to (a + b)$. In an analysis class, this theorem of course depends on a definition for sequence convergence, something equivalent to: *A sequence* (a_n) *converges to a real number* a *if, for every* $\epsilon > 0$*, there exists a natural number* N *such that whenever* $n \geq N$ *it follows that* $|a_n - a| < \epsilon$. Presuming students to be both familiar with this definition of convergence, and also knowledgeable about what it means visually, the proof of the algebraic limit theorem still demands an ability to coordinate and separate multiple uses of this definition on three distinct sequences. For students, not only can the definition of convergence be problematic, but recognizing what is gained when convergence is assumed as well as how to coordinate amongst its various uses is difficult. Again, I do not provide the complete proof in this text but rather elaborate on a few particular aspects. (Note that, similar to zooming, the online version does not allow for one to change the viewing window; downloading the file provides such flexibility.) First, the proof requires understanding the given information, especially as it relates to an ability to choose any particular epsilon value, $\epsilon > 0$, and be guaranteed that at some point the original sequences will permanently enter that ϵ-neighborhood. In the familiar style of an "$\epsilon - N$ challenge," the target ϵ value is only linked to the sequence $(a_n + b_n)$, but otherwise simply represents some specific value that can be manipulated. Second, the use of the triangle inequality in the proof, $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$, conveys a relatively straightforward statement about error: for a sum of two sequences, the worst error is no more than the sum of the errors for the two individual sequences. Third, in addition to having to negotiate various meanings and uses of ϵ in the definition of convergence, the proof also requires coordinating the meaning for two different values for N (i.e., N_1 and N_2), as well as their relationship to the desired value of N.

2.5 An initial look at the dynamic visualization.

The pre-constructed GeoGebra file (https://www.geogebra.org/m/qjd7JJJT) was created so as to mimic the so-called $\epsilon - N$ challenge and response associated with using the definition to prove convergence of the sequence $(a_n + b_n)$. First, the specific givens have been included: the sequences (a_n) and (b_n) (which in this sketch are limited to being functions of n) and their respective limits a and b. Each of these can be modified. The first few terms of each sequence and the differences of their last terms from their respective limit values are depicted, so as to be able to tie into the meaning of the triangle inequality in the proof. Also, note an initial random value for ϵ, which can be changed by either using the slider or, better yet, generating a new random value. The ϵ-neighborhood is shown as associated

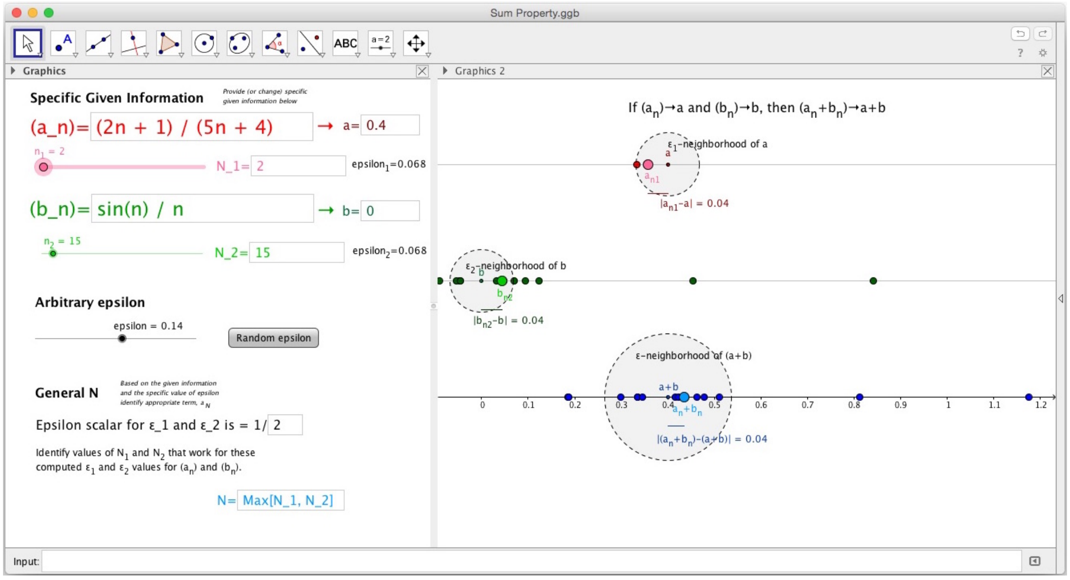

Figure 4: Visualizing the ϵ_1- and ϵ_2-neighborhoods of (a_n) and (b_n)

with the sequence $(a_n + b_n)$. Students are also prompted to consider a scalar for ϵ, which will determine the ϵ_1- and ϵ_2-neighborhoods for the sequences (a_n) and (b_n) respectively — the simplest version is $1/2$, but other scalars might also be acceptable. (Initially, this scalar is undefined, but once it is designated, the two ϵ_1- and ϵ_2-neighborhoods become visualized — see Figure 4.) Second, a brief description of the claim and the goal are described, and sliders for n_1 and n_2 allow students to dynamically visualize the first terms of each sequence. Following the proof, students need to consider a general way, based on what they know about (a_n) and (b_n), to locate a sufficient N for the given ϵ-challenge for the sequence $(a_n + b_n)$. For the scaled ϵ_1- and ϵ_2-neighborhoods, students can physically drag the sliders until the remainder of (a_n) after a term, a_{N_1}, is within the ϵ_1-neighborhood of a and the remainder of (b_n) after a term, b_{N_2}, is within the ϵ_2-neighborhood of b. Initially students can also independently manipulate n; however, for a proof, one would like to determine a sufficient value of N in any situation, based on the values of N_1 and N_2 generated previously - this ends up being, $\max[N_1, N_2]$. By inputting this into the sketch, the slider for n is removed and the first N terms of $(a_n + b_n)$ are automatically generated based on the values of N_1 and N_2 (Figure 4). By updating ϵ for a few specific cases, and manually adjusting N_1 and N_2 (the given information guarantees their existence), students can verify that their term for N is, indeed, sufficient. In fact, if using sequences (a_n) and (b_n) for which students are familiar, one could also determine and input a value of N_1 as a function of ϵ_1, and N_2 as a function of ϵ_2, which would then generalize the proof and update everything accordingly for any newly generated ϵ-challenge.

2.6 Possible explorations to develop proof comprehension through dynamically reconstructed proof visualizations.

As mentioned previously, one of the productive things about a dynamic visualization of proving the sum of two convergent sequences converges by the $\epsilon - N$ definition is that it affords exploring some notable nuances that exist in such arguments. Of course, it would be relatively easy to change the sequences (a_n) and (b_n), and run through the argument; it may also be productive to consider one divergent and one convergent sequence or perhaps even two divergent sequences — a violation of the givens. (One might even consider two divergent sequences whose sum converges, etc. — that is, the converse of the algebraic limit sum theorem is not necessarily true). However, instead of these, which, again, may be worth considering, I focus on two variants that utilize the dynamically reconstructed proof visualization in ways that aim to improve understanding about the proof of this algebraic limit theorem for sequences.

Variant 1. What happens when N_1 and N_2 are different? In fact, this will occur most of the time. However, it is important to be able to reconcile the meaning of what is going on. In particular, the nth term of the sequence $(a_n + b_n)$ is related to the nth terms of the two individual sequences. That is, although one could separately alter the values for n_1 and n_2, they really make the most sense in relation to $(a_n + b_n)$ when they are the same value. In other words, for a given ϵ-challenge, scaled by $1/2$ for each of the original sequences, $N_1 = 2$ may be a sufficient value for the sequence (a_n) but manually setting N_2 and N equal to this value does not result in a proper response for the other two sequences. Whereas if, say, $N_1 = 2$ and $N_2 = 15$ are sufficient values for (a_n) and (b_n), respectively, for a given ϵ-challenge, by setting N_1 and N equal to the larger value, of 15 in this case, it is clear that (a_n) must be within the ϵ_1-neighborhood (since $N_1 = 2$ was sufficient), and that it follows that the sequence $(a_n + b_n)$ must also subsequently be within the desired ϵ-neighborhood. In fact, this brings up the idea that the process used in the proof will not necessarily find the *first* term of $(a_n + b_n)$ for which the sequence enters the ϵ-neighborhood forever, but rather just one that is sufficient. It also suggests that if one wanted to “tinker” with the scalar values for ϵ_1 and ϵ_2 separately, one could find N_1 and N_2 values relatively “close” together so as to locate the first such term in $(a_n + b_n)$; however, although this can be done in any specific case, describing how to do this in a more general case is unnecessarily cumbersome. Thus, the generality of the $\epsilon/2$ scalar for both (a_n) and (b_n) is important not because it will help locate the *first* term for which $(a_n + b_n)$ will be within the desired ϵ-neighborhood, but rather because it will *always* locate a *sufficient* value of N.

Variant 2. What happens if you change the scalar? Although the typical proof uses a scalar of $\epsilon/2$ for (a_n) and (b_n), this need not be the case. One could, for example, use $\epsilon/3$ instead. Essentially, this makes the ϵ_1- and ϵ_2-neighborhoods even smaller, increasing the value(s) of N_1 and N_2. This only further increases the value of N, which will of course also be sufficient for the ϵ-neighborhood of $(a_n + b_n)$. Indeed, the line in the proof only changes slightly, from “$|(a_n + b_n) - (a + b)| < \epsilon/2 + \epsilon/2 = \epsilon$” to “$|(a_n + b_n) - (a + b)| < \epsilon/3 + \epsilon/3 = 2/3\epsilon < \epsilon$.” Interesting, students can also gather further insight by considering scalar values that would not result in a valid proof — for example, not changing the ϵ value at all (scaling by $\epsilon/1$). Evident in Figure 5, both the N_1 and N_2 values are an appropriate response to the (unscaled) ϵ-challenge, but the maximum of these two values is still not a sufficient N-value for $(a_n + b_n)$: the term $(a_N + b_N)$ is outside the ϵ-neighborhood of $a + b$.

The dynamically reconstructed proof visualization, along with both of these variants, has led to important insights about the algebraic limit theorem for the sum of two sequences and its proof. It has been utilized to help clarify some of the various aspects of the definition of convergence — namely, the various ϵs and Ns — and their uses and coordination in the proof. The dynamic nature of a visualized proof is powerful for precisely this reason - that one can visualize the coordination amongst all the various sequences under discussion, which can help foster deeper insight into the process of and logic underlying the proof of the claim.

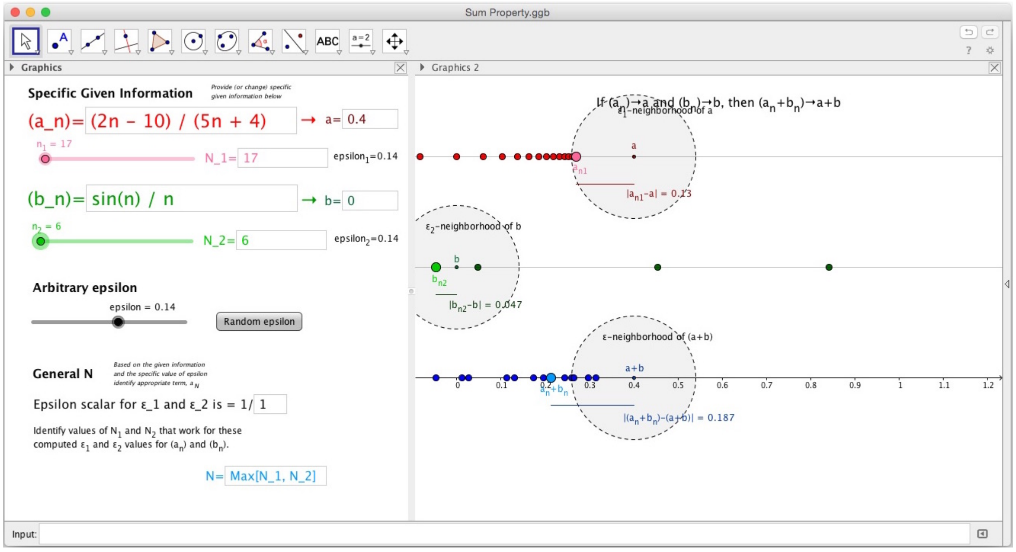

Figure 5: An insufficient ϵ-scalar in the Sum Property of Convergent Sequences

3 Conclusion

Proof is an integral part of a real analysis course - as well as other advanced undergraduate mathematics courses. Evident from these two examples, dynamic visualizations of a proof — which were designed based on adapting Leung's ([4]) model for dynamic task design — can provide opportunities for students to conceptualize some important facets of the proof. First, given the abstract nature of mathematical proofs, dynamic visualizations afford a means to connect the abstract with the specific, providing students with an opportunity to ground some of the abstract qualities of proof in specific situations. In real analysis, the dynamic nature of the technology aligns particularly nicely with the dynamic nature of the abstract objects and proof processes. Second, rather than just a single example on the board or in the textbook, students (and instructors) have some control over the multiple examples one might consider. In particular, I regard this approach of dynamic visualizations as potentially powerful for students' proof comprehension. Although one might be leery about too much reliance upon visuals in the proof process, or the limitations of exploring infinite objects and processes on inherently discrete technologies, the dynamic nature of the visuals capitalizes on the pedagogical potential by improving the variety of possibilities and altering the nature of a student's interactions with the processes and arguments within a proof. Lastly, this approach does not end with the two examples here: one could consider dynamically reconstructing other important proofs in proof-based courses, such as the Bolzano-Weierstrass Theorem, other Algebraic Limit Theorems for sequences, the Fundamental Theorem of Calculus, etc. For those interested, the website, https://www.geogebra.org/m/hqDMd5jN, contains other dynamically reconstructed proof visualizations for use in a real analysis course.

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